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We introduce generalized Husimi functions at the interfaces of dielectric systems. Four different functions can be defined, corresponding to the incident and departing wave on both sides of the interface. These functions allow to identify mechanisms of wave confinement and escape directions in optical microresonators, and give insight into the structure of resonance wave functions. Off resonance, where systematic interference can be neglected, the Husimi functions are related by Snell's law and Fresnel's coefficients.

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Optical microresonators receive growing interest over the last years, because of the intricate interplay of shape (leading to irregular classical ray dynamics), openness of the system (offering means of excitation and escape), and the wave nature of the field. This interplay, together with the promising prospect of applications in future communication devices, has stimulated experiments [1–3] as well as theoretical investigations [4, 5] that were based on concepts well-known from scattering theory, classical ray dynamics, semiclassics, and quantum chaos [6]. A particularly useful tool to study waves in dynamical systems is the Husimi representation of the wave function in classical phase space. So far, the Husimi representation was mostly used to study the closed analogues of optical microsystems, with the dielectric interface being replaced by hard walls, and the principal confinement and radiation directions have been inferred by adding the laws of reflection and refraction by hand. Most notably, efforts in this direction suffer from the fact that the incident and emerging wave components cannot be discriminated by the conventional Husimi representation. The reasons for these fundamental shortcomings arise from the facts that the wave function of the dielectric system is only partially confined by the internal reflection at the refractive index boundary, and that it is affected by the different nature of the boundary conditions, which are neither of Dirichlet nor of von-Neumann type but of a mixed type that follows from Maxwell's equations (both the wave function and its derivative are non-vanishing at the interface).

In this paper we introduce four Husimi functions appropriate for dielectric interfaces, corresponding to the intensity of incident and emerging waves at both sides of the interface. In the regime of ray optics it will be demonstrated that these Husimi functions are related across the interface via Fresnel's formulas, with phase space being deformed according to Snell's law. This connection can be seen as a new variant of the inside-outside duality [7]. However, ray optics only applies when the wave length is short and when systematic interference effects can be neglected. The Husimi functions do

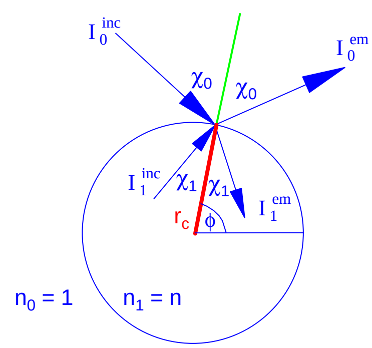


FIG. 1: Refractive-index boundary of a dielectric system.

not require these limitations and develop their full predictive power especially when systematic interference effects lead to strong deviations from Fresnel's and Snell's laws. In particular, ray optics breaks down close to resonances, where the internal part of the scattering wave function is known to be almost independent of the incoming wave that excites the system. The Husimi functions still provide an accurate representation of the wave function (in particular, they nicely display the radiation directions of the field). We illustrate these features using the dielectric circular disk and an annular system as examples [8].

*Microresonators and ray optics.* Consider the light that illuminates and permeates a dielectric system confined by a refractive-index boundary, as shown in Fig. 1. The disk (and also the annulus) are confined by a circular dielectric interface of radius $r_c \equiv 1$. Polar coordinates $r$, $\Phi$ will be used to parameterize position space. We distinguish four wave components: The incident (inc) wave and the emerging (em) wave on both sides of the interface (region 0 with refractive index $n_0 = 1$ outside the microresonator, region 1 with $n_1 \equiv n$ inside the microresonator). In the regime of ray optics, the wave is represented by rays, where the angles of incident and emerging rays are related by the laws of reflection and Snell's law, $n\sin\chi_1 = \sin\chi_0$. For our circular interfaces Snell's law is equivalent to conservation of the angular-momentum variable $m = k_j \sin\chi_j$, where $k_0$ and $k_1 = nk_0$ are the wave number in each region. The ray intensities on either side follow from Fresnel's laws,

$$I_0^{\mathrm{em}} = R_0(\chi_0)\, I_0^{\mathrm{inc}} + T_1(\chi_1)\, I_1^{\mathrm{inc}}, \quad (1)$$
$$I_1^{\mathrm{em}} = R_1(\chi_1)\, I_1^{\mathrm{inc}} + T_0(\chi_0)\, I_0^{\mathrm{inc}}. \quad (2)$$

The reflection and transmission coefficients $R_i$ and $T_i$ are related by

$$R_0(\chi_0) = R_1(\chi_1) \equiv R\,, \quad T_0(\chi_0) = T_1(\chi_1) = 1 - R\,, \quad (3)$$

with [8] $R = \sin^2(\chi_1 - \chi_0)/\sin^2(\chi_1 + \chi_0)$.

*Husimi functions at a dielectric interface.* The Husimi function at the system boundary of closed systems was introduced in Refs. [9, 10] by projection of the conventional Husimi function from full phase space [coordinates $(r, \Phi)$, momentum $(k_j \sin\chi_j, k_j \cos\chi_j)$] onto the reduced phase space at the boundary $r = r_c$ with coordinates $\phi = \Phi$ and $\sin\chi_j$ [11]. The four different Husimi functions (corresponding to the incident and emerging wave at both sides of a dielectric interface) can be constructed by the same procedure when the appropriate boundary conditions are employed. The intensities $I(\phi, \sin\chi) = H(\phi, \sin\chi)\, d\phi\, d\sin\chi$ will turn out to be related by the laws (1), (2) when ray optics applies, but also accurately describe the wave function when ray-optical relations across the interface break down, as expected, e.g., for resonances.

The conventional Husimi function for a given wave function $\Psi(r, \Phi)$ of the dielectric system is obtained as the overlap with a wave packet with minimal uncertainty in the variables $(r, \Phi)$ for real space and $(k_j \sin\chi_j, k_j \cos\chi_j)$ for momentum space. The projection onto the boundary can be formulated rigorously [10]: The wave function is expressed by means of advanced and retarded Green's functions, which in turn allow to distinguish between incident and emerging waves. Green's formula is then used to express the solution $\Psi_j$ of the Helmholtz equation in region $j = 0, 1$ as an integral over the boundary, involving both $\Psi_j$ and its normal (radial) derivative $\Psi_j'$. A semiclassical (saddle-point) approximation then allows to identify in these expressions the following four different Husimi functions on the interface,

$$H_j^{\mathrm{inc(em)}}(\phi, \sin\chi) = \frac{k_j}{2\pi}\left|(-1)^j \mathcal{F}_j h_j(\phi, \sin\chi) + (-)\frac{i}{k_j \mathcal{F}_j} h_j'(\phi, \sin\chi)\right|^2, \quad (4)$$

with the angular-momentum dependent weighting factor $\mathcal{F}_j = \sqrt{n_j \cos\chi_j}$ [12]. Here the functions

$$h_j = \int_0^\infty r\,dr \int_0^{2\pi} d\Phi\, \Psi_j(r, \Phi)\xi(r, \Phi; \phi, \sin\chi), \quad (5)$$
$$h_j' = \int_0^\infty r\,dr \int_0^{2\pi} d\Phi\, \Psi_j'(r, \Phi)\xi(r, \Phi; \phi, \sin\chi) \quad (6)$$

are overlaps with the minimal-uncertainty wave packet

$$\xi(r, \Phi; \phi, \sin\chi) = \sum_l e^{-\frac{1}{2\sigma}(\Phi + 2\pi l - \phi)^2 - ik\sin\chi(\Phi + 2\pi l)} \times (\sigma\pi)^{-\frac{1}{4}}\delta(r - r_c) \quad (7)$$

(a periodic function in $\Phi$), which is restricted to the interface and centered around $(\phi, \sin\chi)$. The parameter $\sigma$ controls its extension in $\phi$-direction, thereby also fixing the uncertainty in $\sin\chi$. We set $\sigma = \sqrt{2}/k_1$. The scaling with $k_1$ results in the same resolution in $\phi$ for all four Husimi functions.

*Inside-outside duality.* As a consequence of the boundary conditions derived from Maxwell's equations we find the identities $h_0(\phi, \sin\chi_0) = h_1(\phi, \sin\chi_1)$, $h_0'(\phi, \sin\chi_0) = h_1'(\phi, \sin\chi_1)$, where the angles $\chi_i$ are related by Snell's law. From these relations it follows that our Husimi functions strictly fulfill the condition of intensity conservation,

$$nH_0^{\mathrm{em}}(\phi, \sin\chi_0) + H_1^{\mathrm{em}}(\phi, \sin\chi_1) = nH_0^{\mathrm{inc}}(\phi, \sin\chi_0) + H_1^{\mathrm{inc}}(\phi, \sin\chi_1)\,, \quad (8)$$

where the factor $n = d\sin\chi_0/d\sin\chi_1$ accounts for the phase-space deformation by Snell's law. Additional relations between the Husimi functions can be anticipated in the regime of ray optics: The intensities on one side of the interface should be related to the intensities on the other side by Eqs. (1), (2). We then expect validity of the resulting inside-outside duality relations

$$H_0^{\mathrm{em}} \approx S(H_0^{\mathrm{em}}) = \frac{1}{n}\frac{1 - 2R}{1 - R}\, H_1^{\mathrm{inc}} + \frac{1}{n}\frac{R}{1 - R}\, H_1^{\mathrm{em}}, \quad (9)$$
$$H_0^{\mathrm{inc}} \approx S(H_0^{\mathrm{inc}}) = -\frac{1}{n}\frac{R}{1 - R}\, H_1^{\mathrm{inc}} + \frac{1}{n}\frac{1}{1 - R}\, H_1^{\mathrm{em}}, \quad (10)$$

which express the Husimi functions in region 0 by the Husimi functions in region 1. The notation $S(\cdot)$ indicates that the approximation is of semiclassical (short wave length) nature; most noticeably, the Husimi functions (intensities) are added incoherently. The duality relations can also be inverted,

$$H_1^{\mathrm{em}} \approx S(H_1^{\mathrm{em}}) = n\frac{1 - 2R}{1 - R}\, H_0^{\mathrm{inc}} + n\frac{R}{1 - R}\, H_0^{\mathrm{em}}, \quad (11)$$
$$H_1^{\mathrm{inc}} \approx S(H_1^{\mathrm{inc}}) = -n\frac{R}{1 - R}\, H_0^{\mathrm{inc}} + n\frac{1}{1 - R}\, H_0^{\mathrm{em}}. \quad (12)$$

However, the Husimi functions in region 1 can only be reconstructed in the strip $|\sin\chi_1| < 1/n$, because the rest of phase space is isolated from region 0 by total internal reflection.

The duality relations are exactly fulfilled in two simple cases, namely, if one incident or emerging wave vanishes or when the two incident waves have the same intensity (the two emerging waves then have the same intensity, as well). We now test the duality relations in more general situations.

The panels on the left in Fig. 2 show the exact Husimi functions from Eq. (4) for the case of the circular disk which is excited by a plane wave at an off-resonant excitation frequency. The illuminating plane wave is clearly visible in the Husimi

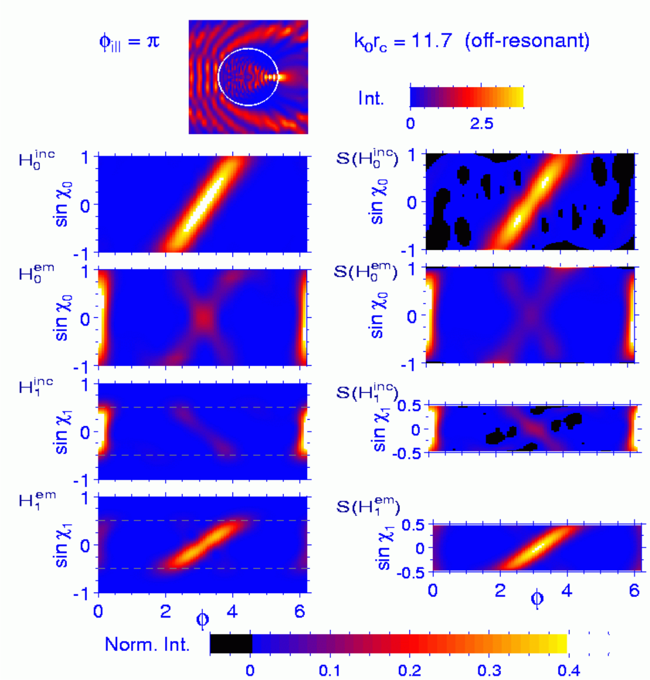


FIG. 2: Exact (left) and reconstructed (right) Husimi functions for the circular dielectric disk ($n = 2$) illuminated by a plane wave at an off-resonant frequency. The top panel shows the scattering wave function in real space. The exact Husimi functions are obtained from Eq. (4). The reconstructed Husimi functions are obtained by Eqs. (9)–(12). Negative Husimi densities are shown in black. The dashed lines in the panels for region 1 mark the critical angle of incidence for total internal reflection.

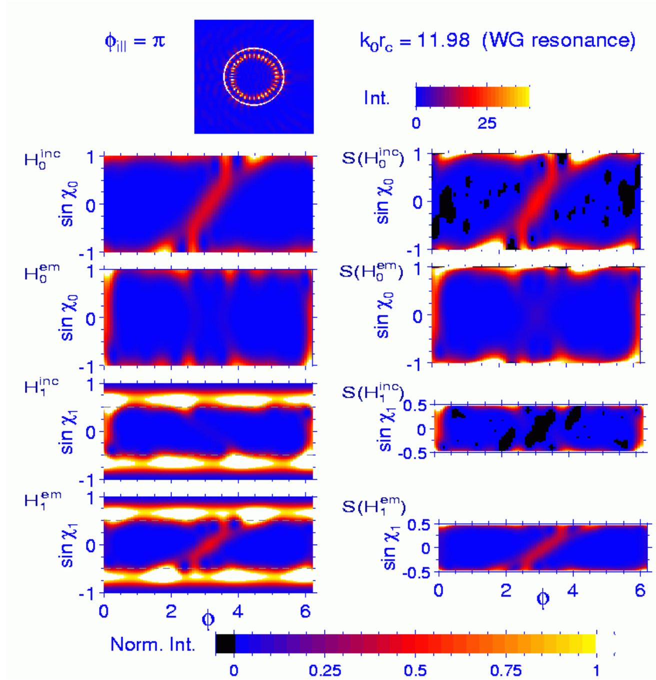


FIG. 3: Same as Fig. 2, but for illumination at a resonance frequency.

function $H_0^{\rm inc}$, around the polar angle $\phi = \phi_{\rm ill} = \pi$, while the focal point of the dielectric disk results in a bright spot in $H_0^{\rm em}$ that is located around $\phi = 0$. There is a close correspondence between the Husimi functions of the incident and emerging waves, and the deformation of phase space by the stretching factor $n$ of Snell's law is clearly visible.

The right panels of Fig. 2 show for comparison the predictions of Eqs. (9)–(12). In the reconstruction we used the slightly modified semiclassical versions of Fresnel's coefficients and Snell's laws given in Ref. [13], which are appropriate for the present case of a curved interface (this results in a slight, but still noticeable quantitative improvement of the reconstruction). We observe a good qualitative and quantitative agreement with the exact Husimi functions. Regions with unphysical negative intensities are small. The most interesting deviations between the exact and the reconstructed Husimi functions occur around the central spot at $\phi = \pi, \sin\chi_j = 0$, where the incoherent predictions of Eqs. (9)–(12) underestimate the exact Husimi densities $H_0^{\rm inc}$, $H_0^{\rm em}$, while they overestimate the intensities $H_1^{\rm inc}$, $H_1^{\rm em}$ in the same area of phase space. These deviations arise from a Fabry-Perot like systematic interference which is constructive in backward direction at the presently chosen frequency. At other frequencies the interference is destructive, and both cases alternate periodically.

*Resonances.* Figure 3 displays the situation for excitation at a frequency which is close to a narrow resonance, a whispering gallery (WG) mode located around $\sin\chi_1 = 0.667$. The top panel shows that the wave function is now well confined inside the disk (region 1). Correspondingly, the Husimi functions $H_1^{\rm inc}$ and $H_1^{\rm em}$ noticeably exceed the Husimi functions $H_0^{\rm inc}$ and $H_0^{\rm em}$. Moreover, the Husimi functions $H_1^{\rm inc}$ and $H_1^{\rm em}$ are dominated by the characteristics of the resonance wave function and consequentially are almost independent of the choice of the exciting wave. (The remnants of the exciting plane can be identified when comparing Fig. 3 with Fig. 2.) Hence the reconstructed Husimi functions deviate noticeably from the exact Husimi functions around $|\sin\chi_1| \sim 1/n$. This is no surprise since resonances are formed by systematic constructive interference, and incoherent ray optics cannot be expected to apply under these circumstances. Most importantly, by principle, the confined wave intensity in the region $|\sin\chi_1| > 1/n$ cannot be reconstructed because classically no refracted rays ever reach this region (which is dark off resonance). On the other hand, the exact Husimi functions display nicely all the features of the resonance wave function in phase space.

Finally, let us illustrate the usefulness of the Husimi functions (4) also for a more complex system than the circular disk, the annular system formed by regions of different refractive indices that are confined by two eccentric disks. The ray optics in this system corresponds to nonintegrable dynamics in phase space, which allows for a multifaceted set of resonance wave functions [5]. Off resonance (Fig. 4) the scattering wave function enters the dielectric system only barely, and the sit-

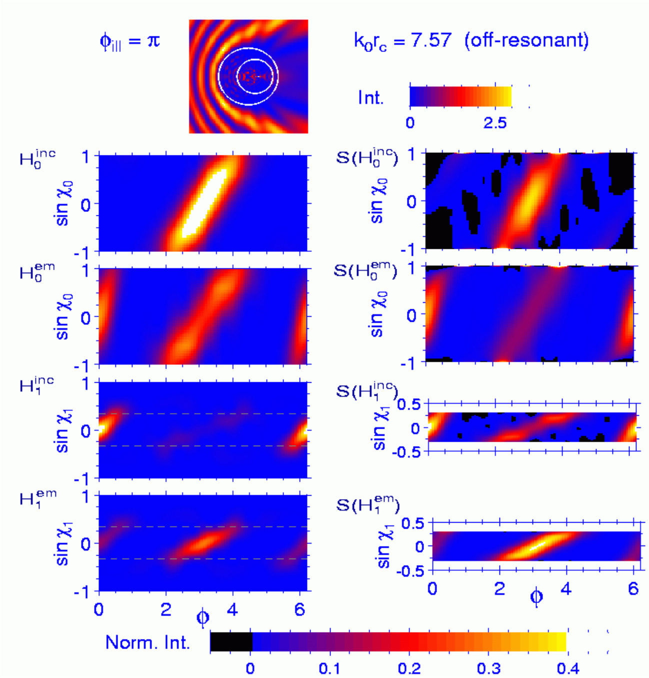


FIG. 4: Same as Fig. 2, but for a dielectric annulus (refractive indices $n_0 = 1$ outside, $n_1 = 3$ in the annulus, $n_2 = 6$ in the inner disk, radii $r_c = 1, r_c' = 0.6$, displaced by $\delta = 0.22$). The system is illuminated by a plane wave at an off-resonant frequency (illumination direction $\phi_{\rm ill} = \pi$).

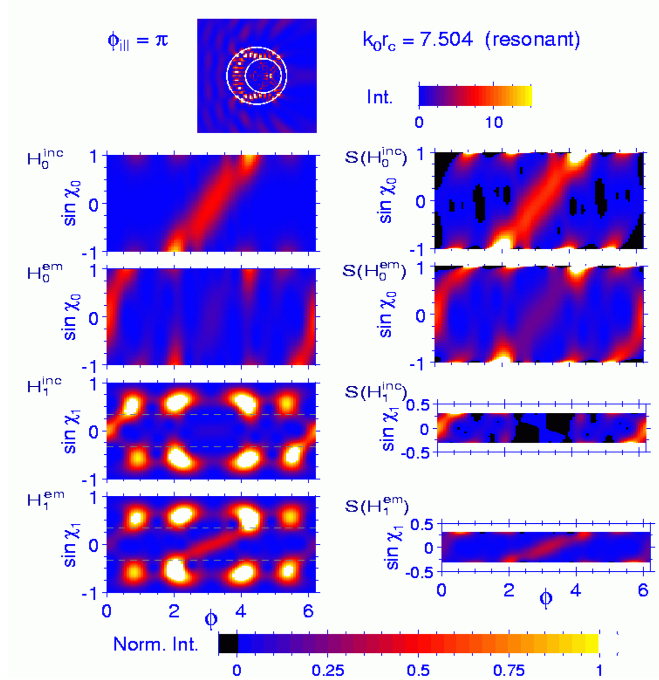


FIG. 5: Same as Fig. 4, but for illumination at a resonance frequency.

uation is similar to the circular disk because the internal disk is not explored extensively. At resonance the situation is very different. Figure 5 shows a typical resonance wave function in real space and its Husimi representation in phase space. The intensity of the resonance wave function is concentrated on straight segments which can be identified as a short stable periodic trajectory of the corresponding classical ray dynamics. The Husimi functions display a strong intensity exactly in the vicinity of this trajectory in classical phase space.

In conclusion, we introduced four Husimi representations of the scattering wave function at the interfaces of dielectric microresonators, corresponding to the incident and emerging waves at both sides of the interface. These Husimi functions are easily computed from the wave function and have many desirable properties: They are related by the laws of Fresnel and Snell in the ray-optics regime (i.e., short and off-resonant wavelength) and generally provide valuable detailed insight into the wave dynamics in complex dielectric systems, most notably even close to resonances where ray optics breaks down.

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